

POINCARÉ POLYNOMIALS OF MODULI SPACES OF STABLE MAPS INTO FLAG MANIFOLDS

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ABSTRACT. By using Bialynicki-Birula decomposition for the stack of genus zero stable maps to flag manifolds[15]. We calculate the Poincaré polynomial of the moduli space in degree one and degree two.

1. INTRODUCTION

In enumerative geometry, when one wants to know about rational curves of degree $\mathbf{d} \in H_2(X)$ in a space X , we consider the space $\text{Mor}_{\mathbf{d}}(\mathbb{P}^1, X)$ of morphisms from \mathbb{P}^1 to X of degree $\mathbf{d} \in H_2(X, \mathbb{Z})$, and use intersection theory on $\text{Mor}_{\mathbf{d}}(\mathbb{P}^1, X)$ to solve the enumerative problem. However, the problem is that the space $\text{Mor}_{\mathbf{d}}(\mathbb{P}^1, X)$ is not compact, so we compactify it. When $X = Fl(r_1, \dots, r_{l+1}; k) = Fl(r_1, \dots, r_{l+1})$ the flag manifold which parametrizes successive subspaces in \mathbb{C}^k :

$$V_1 \subset V_2 \subset \dots \subset V_l \subset \mathbb{C}^k$$

with $\dim V_j = \sum_{i=1}^j r_i$ and $\sum_{i=1}^{l+1} r_i = k$. There is a natural compactification called the hyperquot scheme, it is wildly used in Gromov-Witten theory and quantum cohomology ring of Grassmannian and flag manifolds. In [17] Strømme derive an implicate formula for the Betti numbers of the Quot schemes using Bialynicki-Birula decomposition for Quot scheme. Later in [8], Chen generalized the method to partial flag manifolds and computed the generating function for the Poincaré polynomials of hyperquot schemes.

However there is another natural compactification of $\text{Mor}_{\mathbf{d}}(\mathbb{P}^1, X)$, that is Kontsevich's moduli space of stable maps $\mathcal{M}_0(Fl(r_1, \dots, r_{l+1}), \mathbf{d})$. In [10] Fulton and Pandharipande shows that its coarse moduli space is a projective normal variety with an orbifold structure. In [13], Manin calculate its virtual Poincaré polynomial. Oprea's work [15] shows that there is a Bialynicki-Birula decomposition for the moduli space of stable maps to projective spaces. Applying Oprea's decomposition to the moduli space of stable maps to Grassmannian. Agrawal [1] computes the Euler characteristics of the coarse moduli space of stable maps to Grassmannian in lower degrees and later in [14] Martín computes its Poincaré polynomial in lower degree. Edwards [9] computes the Euler characteristics of the coarse moduli space of stable maps to flag manifolds in lower degrees.

In this paper, we carry out localization analysis on flag manifolds, and compute the Poincaré polynomial of the moduli space of stable maps to flag manifolds in lower degree using the corresponding Bialynicki-Birula decomposition. In our computation, holomorphic Lefschetz formula plays a important role. Our main results are summarized in the following theorem:

Theorem 1. *Let $Fl = Fl(r_1, \dots, r_{l+1})$ be the partial flag manifold, and $\mathcal{M}_0(Fl(r_1, \dots, r_{l+1}), \mathbf{d})$ be the moduli space of genus zero stable maps of degree \mathbf{d} . Then its Poincaré polynomials in degree one and two are:*

$$\begin{aligned} (1) \quad P_{\mathcal{M}_0(Fl(r_1, \dots, r_{l+1}), \tilde{H}_i)}(q) &= \frac{[r_i]_t [r_{i+1}]_t}{1+t} P_{Fl}(q) \\ (2) \quad P_{\mathcal{M}_0(Fl, \tilde{H}_i + \tilde{H}_j)}(q) &= (1+t^2) \frac{[r_i]_t [r_{i+1}]_t [r_j]_t [r_{j+1}]_t}{1+t} P_{Fl}(q), \quad j-i > 1 \\ (3) \quad P_{\mathcal{M}_0(Fl, 2\tilde{H}_i)}(q) &= \frac{(1-t^{r_i})(1-t^{r_{i+1}})((1+t^{r_i+r_{i+1}})(1+t^3) - t(1+t)(t^{r_i} + t^{r_{i+1}}))}{(1-t)^2(1-t^2)^2} P_{Fl}(q) \end{aligned}$$

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where $t = q^2$ and $P_{Fl}(q) = \binom{k}{r_1, \dots, r_{l+1}}_{q^2}$ is the Poincaré polynomial of the partial flag. When the flag is a complete flag, we have:

$$(4) \quad P_{\mathcal{M}_0(Fl, \check{H}_i + \check{H}_{i+1})}(q) = \frac{1 + 2t + 3t^2 + 3t^3 + t^4}{(1+t)(1+t+t^2)} P_{Fl}(q)$$

We make a comment on the Bialynicki-Birula decomposition used in our paper. In [15], Oprea consider Bialynicki-Birula decomposition for Deligne-Mumford stacks. Later in [16] Skowera extended Oprea's result to that any smooth, proper, tame Deligne-Mumford stack, whose coarse moduli space is a scheme admits a Bialynicki-Birula decomposition. He also remarks that (see Remark 3.6) when the coarse moduli space is a projective scheme then the induced decomposition is filterable in the sense of Oprea, and then one may use Lemma 6 in [15] to compute the the Betti numbers of the moduli space from the fixed locus.

This paper is organized as follows: In Section 2, we recall Bialynicki-Birula decomposition and holomorphic Lefschetz formula. In Section 3, we carry out localization analysis on the moduli space. In section 4, we use holomorphic Lefschetz formula to compute the contributions of fixed locus and prove our main theorems.

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2. PRELIMINARIES

2.1. Bialynicki-Birula decomposition. The theory of Bialynicki-Birula decomposition is developed in [4][5] (see also the book [7]). Let X be a smooth projective variety, and T an algebraic torus of dimension one such that T acts on X . Suppose the fixed point set X^T is nontrivial. Let Y_1, \dots, Y_r be the irreducible components of X^T . There is a decomposition of the tangent bundle when restricted to Y_i :

$$(5) \quad TX|_{Y_i} = T_i^+ \oplus T_i^0 \oplus T_i^-$$

where T_i^+ , T_i^0 and T_i^- are subbundles of $TX|_{Y_i}$ such that the torus acts on it with positive, trivial and negative weights respectively. We denote the rank of T_i^+ by p_i and that of T_i^- by n_i . We define:

$$(6) \quad Y_i^+ = \{x \in X \mid \lim_{t \rightarrow 0} t \cdot x \in Y_i\}$$

which are called the *plus cells*;

$$(7) \quad Y_i^- = \{x \in X \mid \lim_{t \rightarrow \infty} t \cdot x \in Y_i\}$$

which are called the *minus cells*. Then we have the *plus decomposition*:

$$(8) \quad X = \cup_{1 \leq i \leq r} Y_i^+$$

and the *minus decomposition*:

$$(9) \quad X = \cup_{1 \leq i \leq r} Y_i^-$$

Then (see [4, Theorem 4.1] or [7, Theorem 4.2]):

- each irreducible component Y_i is smooth and the plus(resp. minus) cells are locally closed;
- the natural projections morphisms $\pi_i^+ : Y_i^+ \rightarrow Y_i$ (resp. $\pi_i^- : Y_i^- \rightarrow Y_i$) are T -isomorphic to $p_i^+ : T_i^+ \rightarrow Y_i$ (resp. $p_i^- : T_i^- \rightarrow Y_i$);
- $T_i^0 \cong TY_i$.

And we may use homology "basis" theorem to compute the homology of X :

Theorem 2 (Homology basis theorem).

$$(10) \quad H_m(X) \cong \oplus_i H_{m-2p_i}(Y_i)$$

$$(11) \quad \cong \oplus_i H_{m-2n_i}(Y_i)$$

So the Poincaré polynomial of the total space X can be computed from that of the fixed locus:

$$(12) \quad P_X(t) = \sum_i t^{2p_i} P_{Y_i}(t) = \sum_i t^{2n_i} P_{Y_i}(t)$$

In many cases, the Y_i 's are isolated points in X . So to compute the Poincaré polynomial, it suffices to determine the numbers p_i or n_i .

2.2. Generalization to Deligne-Mumford stacks. Let \mathcal{M} be a Deligne-Mumford stack with a one-dimensional torus T acting on it. Let \mathcal{F}_i be the components of the fixed locus of the action. Let p_i and n_i be the rank of the corresponding subbundles of the tangent bundle as above. In [15], Oprea proved the Bialynicki-Birula decomposition for Deligne-Mumford stacks provided that there exists a T -equivariant, affine, étale atlas. When the decomposition is filterable, Oprea proves a homology "basis" theorem for Deligne-Mumford stacks:

Theorem 3. *When the decomposition is filterable, the Betti numbers $h^m(\mathcal{M})$ of \mathcal{M} can be computed as*

$$(13) \quad h^i(\mathcal{M}) = \sum_i h^{i-2n_i}(\mathcal{F}_i)$$

Here n_i is the codimension of F_i^+ which equals the number of negative weights on the tangent bundle of \mathcal{M} at a fixed point in \mathcal{F}_i .

Note that the betti numbers are defined using the cohomology theory in [3], and equals the betti number of the coarse moduli space (see [3, Proposition 36]). When the Delign-Mumford stack has a projective coarse moduli space, the decomposition is filterable, then the homology basis theorem of Oprea applies (see [16, Remark 3.6]). This fact is used in [1] and [14] to compute the Euler characteristic and Poincaré polynomial of the moduli space of genus zero stable maps into Grassmannians.

2.3. Holomorphic Lefschetz formula. We will use holomorphic Lefschetz formula to determine the weights of the tangent space to a fixed point. So we recall the holomorphic Lefschetz formula (for details, see [2]):

Let M be a G -manifold, and E be a holomorphic G -vector bundle on M . For any $g \in G$, let M^g be the fixed locus of g , and N^g be the normal bundle. Set

$$(14) \quad \chi_g(M, E) = \sum_q (-1)^q \text{Tr}_g H^q(M, E)$$

then:

$$(15) \quad \chi_g(M, E) = \int_{M^g} \frac{\text{ch}_g(E|_{M^g}) \cdot \text{Tod}(M^g)}{\text{ch}_g \Lambda_{-1}(N^g)^*}$$

where $\Lambda_t(E) = 1 + t\Lambda E + t^2\Lambda^2 E + \dots$ for any vector bundle E , and $\text{ch}_g : K_G(X) \rightarrow H^*(X, \mathbb{C})$ is the homomorphism defined in [2].

3. TORUS ACTION ON THE MODULI SPACE

3.1. Notations. [6] Let $\vec{r} = (r_1, \dots, r_{l+1})$ be an $(l+1)$ -tuple positive integrals with $\sum_i^{l+1} r_i = k$. Let $Fl = Fl(r_1, \dots, r_{l+1}; k) = Fl(r_1, \dots, r_{l+1})$ the flag manifold, which is the moduli space of flags of vector subspaces in \mathbb{C}^k :

$$V_1 \subset V_2 \subset \dots \subset V_l \subset \mathbb{C}^k$$

with $\dim V_j = \sum_{i=1}^j r_i$. We have canonical embedding:

$$(16) \quad Fl(r_1, \dots, r_{l+1}) \hookrightarrow \text{Gr}(s_1, k) \times \dots \times \text{Gr}(s_l, k) \hookrightarrow \mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_l}$$

where $s_i = r_1 + \dots + r_i, i = 1, \dots, l$, and the second arrow is the product of Plücker embeddings. The pull backs of the hyperplane classes $H_i, i = 1, \dots, l$ form a basis of $H^2(Fl, \mathbb{Z})$ and they span the Kähler cone of $Fl(r_1, \dots, r_{l+1})$. Let \tilde{K} be the classes in $H_2(Fl, \mathbb{Z})$ that lies in the dual of the closure of the Kähler cone, and let $\{\tilde{H}_i | i = 1, \dots, l\}$ be the dual basis of $\{H_i | i = 1, \dots, l\}$. Then we may write $\mathbf{d} \in \tilde{K}$ as $\mathbf{d} = d_1 \tilde{H}_1 + \dots + d_l \tilde{H}_l$ with d_i nonnegative integers. Let $\mathcal{M}_0(Fl, \mathbf{d})$ denote the moduli space of genus zero stable maps into $Fl(r_1, \dots, r_{l+1})$ of degree $\mathbf{d} \in \tilde{K}$.

3.2. Torus action on flag manifolds. Let $T = \mathbb{C}^*$ be an algebraic torus. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, $i = 1, \dots, k$ be the canonical basis of \mathbb{C}^k . T acts on \mathbb{C}^k by:

$$(17) \quad T \times \mathbb{C}^k \rightarrow \mathbb{C}^k \\ (t, (x_1, \dots, x_k)) \mapsto (t^{\alpha_1} x_1, \dots, t^{\alpha_k} x_k)$$

where α_i are generic integers such that $\alpha_1 < \alpha_2 < \dots < \alpha_k$. This action induces a torus action on the flag manifold. For convenience, we use the matrix representation of the flag manifold: Let $M_{r,k}^\circ$ be the set of $r \times k$ complex matrices that has r linearly independent rows. For any ${}^t(V_1, \dots, V_r) \in M_{r,k}^\circ$, $V_i \in \mathbb{C}^k$, we associate the flag

$$\text{span}\{V_1, \dots, V_{r_1}\} \subset \text{span}\{V_1, \dots, V_{r_1+r_2}\} \subset \dots \subset \text{span}\{V_1, \dots, V_r\} \subset \mathbb{C}^k$$

in Fl . Thus we have a surjection (in fact a principle bundle):

$$(18) \quad M_{r,k}^\circ \rightarrow Fl(r_1, \dots, r_{l+1})$$

and $Fl(r_1, \dots, r_{l+1})$ may be obtained as the quotient $Gl(r_1, \dots, r_l) \backslash M_{r,k}^\circ$, where $Gl(r_1, \dots, r_l)$ denotes the subgroup of $Gl(r, \mathbb{C})$ that consists of block lower triangular matrices, i.e. invertible matrices of the following form:

$$(19) \quad \begin{pmatrix} A_{r_1} & 0 & \dots & 0 \\ * & A_{r_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & A_{r_l} \end{pmatrix}$$

$Gl(r_1, \dots, r_l)$ acts on $M_{r,k}^\circ$ via left multiplication and T acts in the following manner:

$$(20) \quad \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rk} \end{pmatrix} \cdot t := \begin{pmatrix} t^{\alpha_1} a_{11} & t^{\alpha_2} a_{12} & \dots & t^{\alpha_k} a_{1k} \\ t^{\alpha_1} a_{21} & t^{\alpha_2} a_{22} & \dots & t^{\alpha_k} a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ t^{\alpha_1} a_{r1} & t^{\alpha_2} a_{r2} & \dots & t^{\alpha_k} a_{rk} \end{pmatrix}$$

We easily see from this matrix description that the fixed points of the torus action are $\{P_{I_1, \dots, I_{l+1}} | (I_1, \dots, I_{l+1}) \in \mathfrak{I}(r_1, \dots, r_{l+1})\}$, where the index set $\mathfrak{I}(r_1, \dots, r_{l+1}) = \{(I_1, \dots, I_{l+1}) | I_a \subset \{1, \dots, k\}, |I_a| = r_a, I_a \cap I_b = \emptyset, \forall a \neq b\}$ and $P_{I_1, \dots, I_{l+1}}$ represents the flag:

$$(21) \quad \text{span}\{e_i | i \in I_1\} \subset \text{span}\{e_i | i \in I_1 \cup I_2\} \subset \dots \subset \text{span}\{e_i | i \in I_1 \cup \dots \cup I_l\} \subset \mathbb{C}^k$$

since I_{l+1} is determined by (I_1, \dots, I_l) , we also denote it by $P_{I_1, \dots, I_l; k}$.

Fl is covered by $|\mathfrak{I}(r_1, \dots, r_{l+1})| = \frac{k!}{r_1! \dots r_{l+1}!}$ affine open subsets $\{U_{I_1, \dots, I_{l+1}} | (I_1, \dots, I_{l+1}) \in \mathfrak{I}(r_1, \dots, r_{l+1})\}$, in terms of matrix, $A = (a_{ij}) \in M_{r,k}^\circ$ lies in $U_{I_1, \dots, I_{l+1}}$ if and only if $A_{1, \dots, r_1+ \dots + r_j}^{I_1, \dots, I_j} \neq 0$, $j = 1, \dots, l$, where $A_{i_1, \dots, i_s}^{j_1, \dots, j_s}$ is the minor with column indices $\{j_1, \dots, j_s\}$ and row indices $\{i_1, \dots, i_s\}$:

$$(22) \quad A_{i_1, \dots, i_s}^{j_1, \dots, j_s} = \det A \begin{pmatrix} j_1 & \dots & j_s \\ i_1 & \dots & i_s \end{pmatrix}$$

The matrix representation of a flag in analogy to projective space can be viewed as homogeneous coordinates of the flag manifold. To obtain the inhomogeneous coordinates over the affine open subset $U_{I_1, \dots, I_{l+1}}$, observe that for any $A = (a_{ij}) \in U_{I_1, \dots, I_{l+1}}$, by linear algebra, there exists a unique $g \in Gl(r_1, \dots, r_l)$ such that,

$$(23) \quad A = gB$$

where B is the matrix such that the submatrix $B \begin{pmatrix} I_1, \dots, I_l \\ 1, \dots, r \end{pmatrix}$ is an upper-diagonal matrix with diagonal elements the identities, the matrix B provides the inhomogeneous coordinates. Now let us see how the torus acts on the inhomogeneous coordinates:

$$(24) \quad A \cdot \begin{pmatrix} t^{\alpha_1} & 0 & \dots & 0 \\ 0 & t^{\alpha_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{\alpha_k} \end{pmatrix} = g \cdot \begin{pmatrix} t^{\alpha_{I_1}} & 0 & \dots & 0 \\ 0 & t^{\alpha_{I_2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{\alpha_{I_l}} \end{pmatrix} \cdot \tilde{B}$$

where for a multi-index $I = \{i_1, \dots, i_s\}$, $t^{\alpha_I} = \text{diag}(t^{\alpha_{i_1}}, \dots, t^{\alpha_{i_s}})$, and

$$(25) \quad \tilde{B} = \begin{pmatrix} t^{\alpha_{I_1}} & 0 & \cdots & 0 \\ 0 & t^{\alpha_{I_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{\alpha_{I_l}} \end{pmatrix}^{-1} \cdot B \cdot \begin{pmatrix} t^{\alpha_1} & 0 & \cdots & 0 \\ 0 & t^{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{\alpha_k} \end{pmatrix}$$

Note that the submatrix $\tilde{B}^{(I_1, \dots, I_l)}_{1, \dots, r}$ is still an upper-diagonal matrix with diagonal elements the identities, so \tilde{B} provides the inhomogeneous coordinate of $A \cdot t$. It follows that the tangent space $TFl|_{P_{I_1, \dots, I_{l+1}}}$ at $P_{I_1, \dots, I_{l+1}}$ as a T -representation splits into $\dim Fl$ one-dimensional irreducible representations with weights $\{\alpha_i - \alpha_j | i \notin I_1 \cup \dots \cup I_s, j \in I_s, s = 1, \dots, l\}$.

In summary, we have the following lemma:

Lemma 1. *If we consider $TFl|_{P_{I_1, \dots, I_{l+1}}}$ as a T -representation, the weights are $\{\alpha_i - \alpha_j | i \notin I_1 \cup \dots \cup I_s, j \in I_s, s = 1, \dots, l\}$.*

3.3. Torus action on the moduli space. The above torus action induces a natural torus action on the moduli space of stable maps $\mathcal{M}_0(Fl, \mathbf{d})$ by acting on the target space. To see its fixed points in the moduli space. We need to know the fixed lines in $Fl(r_1, \dots, r_{l+1})$. Since $\chi(\mathbb{P}^1) = 2$, every fixed line in Fl connects two fixed point in Fl . Via the T -equivariant embedding:

$$(26) \quad Fl(r_1, \dots, r_{l+1}; k) \hookrightarrow \text{Gr}(s_1, k) \times \cdots \times \text{Gr}(s_l, k)$$

Every fixed line in Fl is embedded as a fixed line in the product of Grassmannians. Recall the fact that two fixed points $P_{I;k}, P_{J;k}$ of $\text{Gr}(r, k)$ is connected by a fixed line if and only if $|I \cap J| = r - 1$, i.e. the index set I differs from J in only one element (see for example [1]). Explicitly, when $I = \{e_{i_1}, \dots, e_{i_{r-1}}, e_a\}$ and $J = \{e_{i_1}, \dots, e_{i_{r-1}}, e_b\}$, let $P_{I;k} = \text{span}\{e_{i_1}, \dots, e_{i_{r-1}}, e_a\}$ and $P_{J;k} = \text{span}\{e_{i_1}, \dots, e_{i_{r-1}}, e_b\}$ be the corresponding two fixed point in $\text{Gr}(r, k)$, then up to change of coordinate, the fixed line passing through $P_{I;k}$ and $P_{J;k}$ is:

$$(27) \quad \begin{aligned} \mathbb{P}^1 &\hookrightarrow \text{Gr}(r; k) \\ [z : w] &\mapsto \text{span}\{e_{i_1}, \dots, e_{i_{r-1}}, ze_a + we_b\} \end{aligned}$$

Now we analyze the fixed lines in Fl , let $P_{I_1, \dots, I_l; k}$ and $P_{\tilde{I}_1, \dots, \tilde{I}_l; k}$ be two fixed points in $Fl(r_1, \dots, r_{l+1})$ that is connected by a fixed line $\mathbb{P}^1 \hookrightarrow Fl$. Under the embedding (26):

$$(28) \quad P_{I_1, \dots, I_l; k} \mapsto (P_{I_1; k}, \dots, P_{I_1 \cup \dots \cup I_l; k}) \in \text{Gr}(s_1, k) \times \cdots \times \text{Gr}(s_l, k)$$

$$(29) \quad P_{\tilde{I}_1, \dots, \tilde{I}_l; k} \mapsto (P_{\tilde{I}_1; k}, \dots, P_{\tilde{I}_1 \cup \dots \cup \tilde{I}_l; k}) \in \text{Gr}(s_1, k) \times \cdots \times \text{Gr}(s_l, k)$$

and the fixed line is embedded in the product of Grassmannian. When projected to each component $\text{Gr}(s_i, k)$, $i = 1, \dots, l$, the line becomes either a fixed line connecting $P_{I_1 \cup \dots \cup I_s; k}$ and $P_{\tilde{I}_1 \cup \dots \cup \tilde{I}_s; k}$ or a single fixed point.

Let m be the smallest integer such that $P_{I_1 \cup \dots \cup I_i; k} = P_{\tilde{I}_1 \cup \dots \cup \tilde{I}_i; k}$ for $i \leq m$, and $P_{I_1 \cup \dots \cup I_{m+1}; k}$ is connected to $P_{\tilde{I}_1 \cup \dots \cup \tilde{I}_{m+1}; k}$ through a fixed line in $\text{Gr}(s_{m+1}, k)$. Then $I_i = \tilde{I}_i$ for $i \leq m$, and I_{m+1} differs from \tilde{I}_{m+1} in only one element, say $a \in I_{m+1} \setminus \tilde{I}_{m+1}$ and $b \in \tilde{I}_{m+1} \setminus I_{m+1}$. For convenience, we denote I_{m+1} by $A \cup \{a\}$, and \tilde{I}_{m+1} by $A \cup \{b\}$. As for the $(m+2)$ -th component, since $I_1 \cup \dots \cup I_{m+2}$ must be different from $\tilde{I}_1 \cup \dots \cup \tilde{I}_{m+2}$ in at most one element, there are two possibilities: $I_{m+2} = J_{m+2}$ or $A \cup \{a\} \cup I_{m+2} = A \cup \{b\} \cup \tilde{I}_{m+2}$. Let n be the smallest integer such that $I_{m+i+1} = \tilde{I}_{m+i+1}$ for $1 \leq i \leq n$ and $I_{m+1} \cup \dots \cup I_{m+2+n} = \tilde{I}_{m+1} \cup \dots \cup \tilde{I}_{m+2+n}$. For convenience, we denote I_{m+1+i} by J_i , \tilde{I}_{m+1+i} by \tilde{J}_i , $i = 1, \dots, n$, and write I_{m+2+n} as $B \cup \{b\}$, and write \tilde{I}_{m+2+n} as $B \cup \{a\}$. When $i \geq m+3+n$, I_i must be the same as \tilde{I}_i , since otherwise, by a simple argument, the line will not be fixed by the torus action. Again we denote $I_{m+2+n+i}$ by K_i and $\tilde{I}_{m+2+n+i}$ by \tilde{K}_i , $i = 1, \dots, p$, where $n + m + p + 2 = k$. So we may index the set of fixed lines in Fl by the set

$$\mathcal{A} = \bigcup_{m+n+p+2=l+1} \mathcal{A}_{m,n,p}$$

where

$$\begin{aligned}\mathcal{A}_{m,n,p} &= \mathfrak{I}_{r_1, \dots, r_m, r_{m+1}-1, 2, r_{m+2}, \dots, r_{m+n+1}, r_{m+n+2}-1, r_{m+n+3}, \dots, r_{l+1}} \\ &= \{(I_1, \dots, I_m, A, \{a, b\}, J_1, \dots, J_n, B, K_1, \dots, K_p) | \\ &|I_i| = r_i, |A| = r_{m+1} - 1, |J_i| = r_{m+1+i}, |B| = r_{m+2+n} - 1, |K_i| = r_{m+n+2+i}, \\ &\text{the subsets form a partition of the set } \{1, 2, \dots, k\}\}.\end{aligned}$$

Explicitly, for any $(I_1, \dots, I_m, A, \{a, b\}, J_1, \dots, J_n, B, K_1, \dots, K_p) \in \mathcal{A}$, the fixed line is given by:

$$\begin{aligned}[z : w] &\mapsto \text{span}\{e_i | i \in I_1\} \subset \dots \text{span}\{e_i | i \in \cup_{1 \leq j \leq m} I_j\} \\ &\subset \text{span}\{e_i, ze_a + we_b | i \in \cup_{1 \leq j \leq m} I_j \cup A\} \subset \text{span}\{e_i, ze_a + we_b | i \in \cup_{1 \leq j \leq m} I_j \cup A \cup J_1\} \subset \dots \\ &\subset \text{span}\{e_i, ze_a + we_b | i \in \cup_{1 \leq j \leq m} I_j \cup A \cup \cup_{1 \leq j \leq n} J_j\} \\ &\subset \text{span}\{e_i | i \in \cup_{1 \leq j \leq m} I_j \cup A \cup \{a, b\} \cup \cup_{1 \leq j \leq n} J_j \cup B\} \subset \dots \\ (30) \quad &\subset \text{span}\{e_i | i \in \cup_{1 \leq j \leq m} I_j \cup A \cup \{a, b\} \cup \cup_{1 \leq j \leq n} J_j \cup B \cup \cup_{1 \leq j \leq p} K_j\}\end{aligned}$$

which connects the two fixed point $P_{I_1, \dots, I_m, A \cup \{a\}, J_1, \dots, J_n, \{b\} \cup B, K_1, \dots, K_p}$ and $P_{I_1, \dots, I_m, A \cup \{b\}, J_1, \dots, J_n, \{a\} \cup B, K_1, \dots, K_p}$. It is obvious that the fixed line associated to every element in $\mathcal{A}_{m,n,p}$ has degree $\mathbf{d} = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$, where the first m terms and the last $p+1$ terms are zero.

We remark that [9] also contains an analysis of fixed lines in flag manifold by a slightly different argument.

4. COMPUTATION OF POINCARÉ POLYNOMIALS

4.1. Notations. In this section, we always assume that the weights $\alpha_1, \dots, \alpha_{l+1}$ satisfy: $\alpha_i \gg \sum_{j < i} \alpha_j$. Let $\mathcal{A}_n = \{1, 2, \dots, n\}$, and let $\mathcal{A}_{n,j}$ be the collection of all subsets of \mathcal{A}_n with j elements, $1 \leq j \leq n$. For any $(I_1, \dots, I_{l+1}) \in \mathfrak{I}_{r_1, \dots, r_{l+1}}$, we define the representation $V_{I_1, \dots, I_{l+1}}$ as the direct sum of $V_{\alpha_\mu - \alpha_\nu, \mu \notin I_1 \cup \dots \cup I_s, \nu \in I_s, s = 1, \dots, l}$, where V_α is the one-dimensional representation with weight $\alpha \in \mathbb{Z}$. By Lemma 1 we know that $TFl|_{P_{I_1, \dots, I_{l+1}}} = V_{I_1, \dots, I_{l+1}}$. We also define the number

$$N_{I_1, \dots, I_{l+1}} = N_{I_1, \dots, I_l; k} = \sum_{1 \leq s \leq l} \#\{(i, j) | i \notin I_1 \cup \dots \cup I_s, j \in I_s, i > j\}$$

, which is the number of positive weights in $V_{I_1, \dots, I_{l+1}}$.

Note that for any $j_1, j_2, \dots, j_{s+1} \in \mathbb{Z}_+$ such that $j_1 + j_2 + \dots + j_{s+1} = l+1$, we have a map $\mathfrak{I}_{r_1, \dots, r_{l+1}} \rightarrow \mathfrak{I}_{r_1 + \dots + r_{j_1}, r_{j_1+1} + \dots + r_{j_1+j_2}, \dots, r_{j_s+1} + \dots + r_{j_s+j_{s+1}}}$ sending (I_1, \dots, I_{l+1}) to $(\cup_{1 \leq i \leq j_1} I_i, \dots, \cup_{j_s+1 \leq i \leq j_s+j_{s+1}} I_i)$. This is a fibration with fiber $\mathfrak{I}_{r_1, \dots, r_{j_1}} \times \dots \times \mathfrak{I}_{r_{j_s+1}, \dots, r_{j_s+j_{s+1}}}$. Hence, we have isomorphism between sets:

$$(31) \quad \mathfrak{I}_{r_1, \dots, r_{l+1}} \cong \mathfrak{I}_{r_1 + \dots + r_{j_1}, r_{j_1+1} + \dots + r_{j_1+j_2}, \dots, r_{j_s+1} + \dots + r_{j_s+j_{s+1}}} \times (\mathfrak{I}_{r_1, \dots, r_{j_1}} \times \dots \times \mathfrak{I}_{r_{j_s+1}, \dots, r_{j_s+j_{s+1}}})$$

Let $((J_1, \dots, J_{s+1}), (J_{1,1}, \dots, J_{1,j_1}), \dots, (J_{s,1}, \dots, J_{s,j_{s+1}}))$ be the element belonging to the right hand side corresponding to (I_1, \dots, I_{l+1}) under this isomorphism, one can easily see that:

$$(32) \quad N_{I_1, \dots, I_{l+1}} = N_{J_1, \dots, J_{s+1}} + N_{J_{1,1}, \dots, J_{1,j_1}} + \dots + N_{J_{s,1}, \dots, J_{s,j_{s+1}}}$$

This fibration is very useful in our computation, as an example, we calculate

$$f_{r_1, \dots, r_{l+1}}(t) = f_{r_1, \dots, r_l; k}(t) = \sum_{(I_1, \dots, I_{l+1}) \in \mathfrak{I}_{r_1, \dots, r_{l+1}}} t^{N_{I_1, \dots, I_{l+1}}}.$$

in fact, if we take a specific fibration $\mathfrak{I}_{r_1, \dots, r_{l+1}} \rightarrow \mathfrak{I}_{r_1 + \dots + r_l, r_{l+1}}$, using the corresponding isomorphism (31), we have:

$$(33) \quad f_{r_1, \dots, r_{l+1}}(t) = f_{r_1 + \dots + r_l, r_{l+1}}(t) f_{r_1, \dots, r_l}(t)$$

using this equation inductively, we finally have:

$$(34) \quad f_{r_1, \dots, r_{l+1}}(t) = f_{r_1 + \dots + r_l, r_{l+1}}(t) f_{r_1 + \dots + r_{l-1}, r_l}(t) \cdots f_{r_1, r_2}(t)$$

4.2. q -binomials. We recall the concept of q -binomials, we refer the reader to the book [12] for an beautiful exposition. For any positive integer n , the q -number of n is denoted by $[n]_q := \frac{q^n - 1}{q - 1}$; the q -factorial is defined by $[n]_q! := [n]_q [n-1]_q \cdots [1]_q$; and the q -binomial $\binom{n}{k}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!}$. We include here some basic identities:

$$(35) \quad \binom{n}{k}_q = \binom{n}{n-k}_q$$

$$(36) \quad \binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q$$

$$(37) \quad \binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

$$(38) \quad \sum_{j=0}^a q^j \binom{d+j}{j}_q = \binom{d+a+1}{a}_q$$

The following identity can be found in the appendix of [14]:

$$(39) \quad \sum_{i+j=u} t^{i(j'+1)} \binom{i+i'}{i}_t \binom{j+j'}{j}_t = \binom{i'+j'+u+1}{u}_t$$

Using this, we have:

$$(40) \quad \begin{aligned} \sum_{i+j=u} t^{i(j'+2)} \binom{i+i'}{i}_t \binom{j+j'}{j}_t &= \sum_{i+j=u} t^{i((j'+1)+1)} \binom{i+i'}{i}_t \left(\binom{j+j'+1}{j}_t - t^{j'+1} \binom{j+j'}{j-1}_t \right) \\ &= \binom{i'+j'+u+2}{u}_t - t^{j'+1} \sum_{i+j-1=u-1} t^{i((j'+1)+1)} \binom{i+i'}{i}_t \binom{j-1+(j'+1)}{j-1}_t \\ &= \binom{i'+j'+u+2}{u}_t - t^{j'+1} \binom{i'+j'+u+1}{u-1}_t \end{aligned}$$

We have the following combinatoric interpretation of q -binomials:

Theorem 4 ([12]).

$$(41) \quad \binom{n}{j}_q = \sum_{S \in \mathcal{A}_{n,j}} q^{\omega(S) - j(j+1)/2}$$

where $\omega(S) = \sum_{s \in S} s$.

4.3. Poincaré polynomial of $Fl(r_1, \dots, r_{l+1}; k)$. As a warm up, we compute the Poincaré polynomial of the flag manifold itself. By Lemma 1, the number of positive weights at the fixed point $P_{I_1, \dots, I_l; k}$ is $N_{I_1, \dots, I_l; k}$. By (12), the Poincaré of Fl is:

$$(42) \quad P_{Fl}(q) = \sum_{(I_1, \dots, I_{l+1}) \in \mathfrak{I}} q^{2N_{I_1, \dots, I_{l+1}}}$$

The following lemma can be checked by direct computation:

Lemma 2. For any $S \in \mathcal{A}_{k;r}$, we have

$$(43) \quad N_{S;k} = \sum_{i=1}^l \omega({}^t S) - r(r+1)/2$$

where $S \in \mathcal{A}_{n,j} \mapsto {}^t S \in \mathcal{A}_{n,j}$ is the one-to-one map that maps (a_1, \dots, a_j) to $(n+1-a_j, \dots, n+1-a_1)$.

Now, by the lemma and Theorem 4, combining 34 and (42), we can compute the Poincaré polynomial of $Fl(r_1, \dots, r_{l+1})$:

$$\begin{aligned} P_{Fl}(q) &= f_{r_1, \dots, r_l; k}(q^2) \\ &= f_{r_1; k}(q^2) f_{r_2; k-r_1}(q^2) \cdots f_{r_l; k-r_1-\dots-r_{l-1}}(q^2) \\ &= \binom{k}{r_1}_{q^2} \binom{k-r_1}{r_2}_{q^2} \cdots \binom{k-r_1-\dots-r_{l-1}}{r_l}_{q^2} \\ &= \binom{k}{r_1 \cdots r_{l+1}}_{q^2} \end{aligned}$$

4.4. Poincaré polynomial of $\mathcal{M}_0(Fl, \check{H}_i)$. By Theorem 3, it suffices to compute the number of positive (or negative) weights of the tangent space at the fixed points of $\mathcal{M}_0(Fl, \check{H}_i)$. By the analysis in the last section, the fixed point set in $\mathcal{M}_0(Fl, \check{H}_i)$ consists of fixed lines $\mathbb{P}^1 \rightarrow Fl(r_1, \dots, r_{l+1}; k)$ that are parameterized by the index set $\mathcal{A}_{i-1, 0, l-i}$. For any $(I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{l-i}) \in \mathcal{A}_{i-1, 0, l-i}$, let $f : \mathbb{P}^1 \rightarrow Fl(r_1, \dots, r_{l+1}; k)$ be the corresponding fixed line connecting $p_a = P_{I_1, \dots, I_{i-1}, A \cup \{a\}, B \cup \{b\}, K_1, \dots, K_{l-i}}$ and $p_b = P_{I_1, \dots, I_{i-1}, A \cup \{b\}, B \cup \{a\}, K_1, \dots, K_{l-i}}$.

Recall that since the flag manifold is convex, the moduli space is unobstructed, and the tangent space to any fixed point $(\Sigma, f : \Sigma \rightarrow Fl)$ can be identified with $\text{Def}(\Sigma, f)$, which as a T -representation fits into the deformation long exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Aut}(\Sigma) & \rightarrow & \text{Def}(f) & \rightarrow & \text{Def}(\Sigma, f) \\ & & & & \rightarrow & & 0. \end{array}$$

where $\text{Def}(f) = H^0(\Sigma, f^*TFl)$. Apply this long exact sequence to our case (see [11] for analysis of stable maps to \mathbb{P}^n), $\text{Def}(\Sigma) = 0$ and $\text{Aut}(\Sigma) = V_0 + V_{\alpha_a - \alpha_b} + V_{\alpha_b - \alpha_a}$, where V_n is the one-dimensional representation with weight $n \in \mathbb{Z}$. To compute weights of the representation $H^0(\Sigma, f^*TFl)$, note that $H^i(\Sigma, f^*TFl) = 0, i \geq 1$, so $\text{Tr}_g H^0(\Sigma, f^*TFl) = \chi_g(\mathbb{P}^1, f^*TFl)$, and we may use the holomorphic Lefschetz formula to compute it:

$$\begin{aligned} \chi_g(\mathbb{P}^1, f^*TFl) &= \frac{\text{Tr}_g(TFl|_{p_a})}{1 - g^{\alpha_a - \alpha_b}} + \frac{\text{Tr}_g(TFl|_{p_b})}{1 - g^{\alpha_b - \alpha_a}} \\ &= \left[\left(\frac{\sum_{1 \leq h \leq i-1} \sum_{\nu \in I_h, \mu \notin I_1 \cup \dots \cup I_h} g^{\alpha_\mu - \alpha_\nu}}{1 - g^{\alpha_a - \alpha_b}} + \frac{\sum_{1 \leq h \leq i-1} \sum_{\nu \in I_h, \mu \notin I_1 \cup \dots \cup I_h} g^{\alpha_\mu - \alpha_\nu}}{1 - g^{\alpha_b - \alpha_a}} \right) \right] \\ &\quad + \left[\left(\frac{\sum_{\mu \in B \cup K_1 \cup \dots \cup K_{l-i}, \nu \in A} g^{\alpha_\mu - \alpha_\nu}}{1 - g^{\alpha_a - \alpha_b}} + \frac{\sum_{\mu \in B \cup K_1 \cup \dots \cup K_{l-i}, \nu \in A} g^{\alpha_\mu - \alpha_\nu}}{1 - g^{\alpha_b - \alpha_a}} \right) \right] \\ &\quad + \left(\frac{\sum_{\nu \in A} g^{\alpha_b - \alpha_\nu}}{1 - g^{\alpha_a - \alpha_b}} + \frac{\sum_{\nu \in A} g^{\alpha_a - \alpha_\nu}}{1 - g^{\alpha_b - \alpha_a}} \right) \\ &\quad + \left(\frac{g^{\alpha_b - \alpha_a}}{1 - g^{\alpha_a - \alpha_b}} + \frac{g^{\alpha_a - \alpha_b}}{1 - g^{\alpha_b - \alpha_a}} \right) \\ &\quad + \left(\frac{\sum_{\mu \in B \cup K_1 \cup \dots \cup K_{l-i}} g^{\alpha_\mu - \alpha_a}}{1 - g^{\alpha_a - \alpha_b}} + \frac{\sum_{\mu \in B \cup K_1 \cup \dots \cup K_{l-i}} g^{\alpha_\mu - \alpha_b}}{1 - g^{\alpha_b - \alpha_a}} \right) \\ &\quad + \left[\left(\frac{\sum_{\mu \in K_1 \cup \dots \cup K_{l-i}, \nu \in B} g^{\alpha_\mu - \alpha_\nu}}{1 - g^{\alpha_a - \alpha_b}} + \frac{\sum_{\mu \in K_1 \cup \dots \cup K_{l-i}, \nu \in B} g^{\alpha_\mu - \alpha_\nu}}{1 - g^{\alpha_b - \alpha_a}} \right) \right] \\ &\quad + \left(\frac{\sum_{\mu \in K_1 \cup \dots \cup K_{l-i}} g^{\alpha_\mu - \alpha_b}}{1 - g^{\alpha_a - \alpha_b}} + \frac{\sum_{\mu \in K_1 \cup \dots \cup K_{l-i}} g^{\alpha_\mu - \alpha_a}}{1 - g^{\alpha_b - \alpha_a}} \right) \\ (44) \quad &\quad + \left[\left(\frac{\sum_{1 \leq h \leq l-i} \sum_{\nu \in K_h, \mu \in K_{h+1} \cup \dots \cup K_{l-i}} g^{\alpha_\mu - \alpha_\nu}}{1 - g^{\alpha_a - \alpha_b}} + \frac{\sum_{1 \leq h \leq l-i} \sum_{\nu \in K_h, \mu \in K_{h+1} \cup \dots \cup K_{l-i}} g^{\alpha_\mu - \alpha_\nu}}{1 - g^{\alpha_b - \alpha_a}} \right) \right] \end{aligned}$$

in which we use Lemma 1. Observe the following identities:

$$\begin{aligned} \frac{1}{1-z} + \frac{1}{1-z^{-1}} &= 1, & \frac{1}{1-z} + \frac{z}{1-z^{-1}} &= 1+z; \\ \frac{z^{-1}}{1-z} + \frac{z}{1-z^{-1}} &= z^{-1} + 1 + z, & \frac{1}{1-z} + \frac{z^{-1}}{1-z^{-1}} &= 0 \end{aligned}$$

we may continue the computation:

$$\begin{aligned}
\chi_g(\mathbb{P}^1, f^* TFl) = & [(\sum_{1 \leq h \leq i-1} \sum_{\nu \in I_h, \mu \notin I_1 \cup \dots \cup I_h} g^{\alpha_\mu - \alpha_\nu})] + [(\sum_{\mu \in B \cup K_1 \cup \dots \cup K_{l-i}, \nu \in A} g^{\alpha_\mu - \alpha_\nu}) \\
& + (\sum_{\nu \in A} (g^{\alpha_a - \alpha_\nu} + g^{\alpha_b - \alpha_\nu})) + (g^{\alpha_b - \alpha_a} + 1 + g^{\alpha_a - \alpha_b}) + (\sum_{\mu \in B \cup K_1 \cup \dots \cup K_{l-i}} g^{\alpha_\mu - \alpha_a} + g^{\alpha_\mu - \alpha_b})] \\
(45) \quad & + [(\sum_{\mu \in K_1 \cup \dots \cup K_{l-i}, \nu \in B} g^{\alpha_\mu - \alpha_\nu}) + 0] + [(\sum_{1 \leq h \leq l-i} \sum_{\nu \in K_h, \mu \in K_{h+1} \cup \dots \cup K_{l-i}} g^{\alpha_\mu - \alpha_\nu})]
\end{aligned}$$

So as a representation,

$$\begin{aligned}
\text{Def}(\Sigma, f) = & [(\sum_{1 \leq h \leq i-1} \sum_{\nu \in I_h, \mu \notin I_1 \cup \dots \cup I_h} V_{\alpha_\mu - \alpha_\nu})] + [(\sum_{\mu \in B \cup K_1 \cup \dots \cup K_{l-i}, \nu \in A} V_{\alpha_\mu - \alpha_\nu}) \\
& + (\sum_{\nu \in A} (V_{\alpha_a - \alpha_\nu} + V_{\alpha_b - \alpha_\nu})) + (\sum_{\mu \in B \cup K_1 \cup \dots \cup K_{l-i}} V_{\alpha_\mu - \alpha_a} + V_{\alpha_\mu - \alpha_b})] \\
& + (\sum_{\mu \in K_1 \cup \dots \cup K_{l-i}, \nu \in B} V_{\alpha_\mu - \alpha_\nu}) + (\sum_{1 \leq h \leq l-i} \sum_{\nu \in K_h, \mu \in K_{h+1} \cup \dots \cup K_{l-i}} V_{\alpha_\mu - \alpha_\nu}) \\
(46) \quad & = V_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{l-i}}
\end{aligned}$$

So the Poincaré polynomial of $\mathcal{M}_0(Fl, \check{H}_i)$ is:

$$(47) \quad P_{\mathcal{M}_0(Fl, \check{H}_i)}(q) = \sum_{(I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{l-i}) \in \mathcal{A}_{i-1, 0, l-i}} q^{2N_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{l-i}}}$$

$$(48) \quad = \binom{k}{r_1, \dots, r_{i-1}, r_i - 1, 2, r_{i+1} - 1, r_{i+2}, \dots, r_{l+1}}_{q^2}$$

$$(49) \quad = \frac{[r_i]_{q^2} [r_{i+1}]_{q^2}}{1 + q^2} \frac{[k]_{q^2}}{[r_1]_{q^2}! \dots [r_{l+1}]_{q^2}!}$$

$$(50) \quad = \frac{[r_i]_{q^2} [r_{i+1}]_{q^2}}{1 + q^2} P_{Fl}(q)$$

4.5. Poincaré polynomial of $\mathcal{M}_0(Fl, \check{H}_i + \check{H}_j), j - i > 1$. In this case the fixed point set of the torus action consists of maps of the form $\Sigma \rightarrow Fl(r_1, \dots, r_{l+1}; k)$ where Σ is a nodal curve consists of two rational components such that each component is embedded in Fl as a fixed line of homology class H_i and H_j respectively. These fixed maps are parameterized by the set $\mathcal{I}_{i,j}$, where

$$\begin{aligned}
\mathcal{I}_{i,j} = & \{(I_1, \dots, I_{i-1}, A, a, b, B, K_1, \dots, K_{j-i-2}, C, c, d, D, K_{j-i+1}, \dots, K_{l-i}) \mid \\
& |I_j| = r_j, |A| = r_i - 1, |B| = r_{i+1} - 1, |C| = r_j - 1, |D| = r_{j+1} - 1, |K_j| = r_{i+j+1}, \\
& \text{the subsets form a partition of the set } \{1, 2, \dots, k\}\}.
\end{aligned}$$

For every $(I_1, \dots, I_{i-1}, A, a, b, B, K_1, \dots, K_{j-i-2}, C, c, d, D, K_{j-i+1}, \dots, K_{l-i}) \in \mathcal{I}_{i,j}$, the corresponding fixed map $f : \Sigma \rightarrow Fl$ can be described as follows:

- $f(p) = P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, K_1, \dots, K_{j-i-2}, C \cup \{c\}, \{d\} \cup D, K_{j-i+1}, \dots, K_{l-i}}$, where p is the node of Σ .
- one of the two components of Σ , say Σ_1 , is mapped to the fixed line connecting $P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, K_1, \dots}$ and $P_{I_1, \dots, I_{i-1}, A \cup \{b\}, \{a\} \cup B, K_1, \dots}$
- the other component of Σ , say Σ_2 , is mapped to the fixed line connecting $P_{\dots, C \cup \{c\}, \{d\} \cup D, K_{j-i+1}, \dots, K_{l-i}}$ and $P_{\dots, C \cup \{d\}, \{c\} \cup D, K_{j-i+1}, \dots, K_{l-i}}$

we again use the deformation long exact sequence, in this case $\text{Def}(\Sigma) = V_{\alpha_b - \alpha_a + \alpha_d - \alpha_c}$ which corresponds to the one-dimensional deformation of the node, and $\text{Aut}(\Sigma) = V_{\alpha_a - \alpha_b} + V_0 + V_0 + V_{\alpha_c - \alpha_d}$, to calculate $\text{Def}(f)$, we use the normalization exact sequence, let $C = \mathbb{P}^1 \sqcup \mathbb{P}^1 \rightarrow \Sigma$ be the normalization of Σ , then we have:

$$(51) \quad 0 \rightarrow \mathcal{O}_\Sigma \rightarrow \mathcal{O}_C \rightarrow \mathbb{C}_p \rightarrow 0$$

where \mathbb{C}_p is the skyscraper sheaf at the nodal point p . Tensoring with f^*TFl , we have the long exact sequence for cohomology:

$$0 \rightarrow H^0(\Sigma, f^*TFl) \rightarrow H^0(\Sigma_1, f^*TFl) \oplus H^0(\Sigma_2, f^*TFl) \rightarrow TFl|_{f(p)} \rightarrow 0$$

we again use holomorphic Lefschetz formula to compute $H^0(\Sigma_1, f^*TFl)$ and $H^0(\Sigma_2, f^*TFl)$:

$$\begin{aligned} H^0(\Sigma_1, f^*TFl) &= V_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{j-i-2}, C \cup \{c\}, \{d\} \cup D, K_{j-i+1}, \dots, K_{l-i}} + V_0 + V_{\alpha_a - \alpha_b} + V_{\alpha_b - \alpha_a} \\ &= V_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{j-i-2}, C, \{c, d\}, D, K_{j-i+1}, \dots, K_{l-i}} + V_{\alpha_d - \alpha_c} - V_{C, \{c\}} - V_{\{d\}, D} \\ (52) \quad &+ V_0 + V_{\alpha_a - \alpha_b} + V_{\alpha_b - \alpha_a} \end{aligned}$$

$$\begin{aligned} H^0(\Sigma_2, f^*TFl) &= V_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, K_1, \dots, K_{j-i-2}, C, \{c, d\}, D, K_{j-i+1}, \dots, K_{l-i}} + V_0 + V_{\alpha_c - \alpha_d} + V_{\alpha_d - \alpha_c} \\ &= V_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{j-i-2}, C, \{c, d\}, D, K_{j-i+1}, \dots, K_{l-i}} + V_{\alpha_b - \alpha_a} - V_{A, \{a\}} - V_{\{b\}, B} \\ (53) \quad &+ V_0 + V_{\alpha_c - \alpha_d} + V_{\alpha_d - \alpha_c} \end{aligned}$$

and

$$(54) \quad TFl|_{f(p)} = V_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, K_1, \dots, K_{j-i-2}, C \cup \{c\}, \{d\} \cup D, K_{j-i+1}, \dots, K_{l-i}}$$

Hence

$$(55) \quad \text{Def}(\Sigma, f) = \text{Def}(f) + \text{Def}(\Sigma) - \text{Aut}(\Sigma) = V_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{j-i-2}, C, \{c, d\}, D, K_{j-i+1}, \dots, K_{l-i}} + V_{\alpha_b - \alpha_a} + V_{\alpha_d - \alpha_c} + V_{\alpha_b - \alpha_a + \alpha_d - \alpha_c}$$

Note that by our assumption on the weights α_i , $\alpha_b - \alpha_a + \alpha_d - \alpha_c > 0$ if $b = \max\{a, b, c, d\}$ or $d = \max\{a, b, c, d\}$; and $\alpha_b - \alpha_a + \alpha_d - \alpha_c < 0$ if $a = \max\{a, b, c, d\}$ or $c = \max\{a, b, c, d\}$. So the Poincaré polynomial is:

$$\begin{aligned} P_{\mathcal{M}_0(Fl, \check{H}_i + \check{H}_j)}(q) &= (1 + q^2 + q^4 + q^6) \sum_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{j-i-2}, C, \{c, d\}, D, K_{j-i+1}, \dots, K_{l-i}} \\ &\quad q^{2N_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_1, \dots, K_{j-i-2}, C, \{c, d\}, D, K_{j-i+1}, \dots, K_{l-i}}} \\ &= (1 + q^2 + q^4 + q^6) \binom{k}{\dots, r_{i-1}, r_i - 1, 2, r_{i+1} - 1, r_{i+2}, \dots, r_{j-1}, r_j - 1, 2, r_{j+1} - 1, r_{j+2}, \dots}_{q^2} \\ &= (1 + q^2 + q^4 + q^6) \frac{[r_i]_{q^2} [r_{i+1}]_{q^2} [r_j]_{q^2} [r_{j+1}]_{q^2}}{(1 + q^2)^2} \binom{k}{r_1, \dots, r_{l+1}}_{q^2} \\ (56) \quad &= (1 + q^4) \frac{[r_i]_{q^2} [r_{i+1}]_{q^2} [r_j]_{q^2} [r_{j+1}]_{q^2}}{1 + q^2} P_{Fl}(q) \end{aligned}$$

4.6. Poincaré polynomial of $\mathcal{M}_0(Fl, 2\check{H}_i)$. In this case the fixed point set of the torus action is still discrete. And it can be divided into five disjoint subsets which are indexed by $\mathcal{I}_1, \dots, \mathcal{I}_5$ respectively, where

$$\mathcal{I}_1 = \mathcal{J}_{r_1, \dots, r_{i-1}, r_i - 1, 2, r_{i+1} - 1, r_{i+2}, \dots, r_{l+1}}$$

$$\mathcal{I}_2 = \mathcal{J}_{r_1, \dots, r_{i-1}, r_i - 1, 1, 1, r_{i+1} - 1, r_{i+2}, \dots, r_{l+1}}$$

$$\mathcal{I}_3 = \mathcal{J}_{r_1, \dots, r_{i-1}, r_i - 1, 1, 2, r_{i+1} - 2, r_{i+2}, \dots, r_{l+1}}$$

$$\mathcal{I}_4 = \mathcal{J}_{r_1, \dots, r_{i-1}, r_i - 2, 2, 1, r_{i+1} - 1, r_{i+2}, \dots, r_{l+1}}$$

$$\mathcal{I}_5 = \mathcal{J}_{r_1, \dots, r_{i-1}, r_i - 2, 2, 1, 1, r_{i+1} - 2, r_{i+2}, \dots, r_{l+1}}$$

For any $(I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_2, \dots, K_{l-i}) \in \mathcal{I}_1$, it parameterizes a fixed map $f : \Sigma \rightarrow Fl$ described as:

- Σ is isomorphic to \mathbb{P}^1 ;
- f is a covering of degree two from \mathbb{P}^1 to the fixed line connecting $P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, K_2, \dots, K_{l-i}; k}$ and $P_{I_1, \dots, I_{i-1}, A \cup \{b\}, \{a\} \cup B, K_2, \dots, K_{l-i}; k}$

By holomorphic Lefschetz formula, one can check that:

$$(57) \quad \text{Def}(\Sigma, f) = V_{I_1, \dots, I_{i-1}, A, \{a, b\}, B, K_2, \dots, K_{l-i}} + \sum_{\nu \in A} V_{\frac{\alpha_a + \alpha_b}{2} - \alpha_\nu} + \sum_{\mu \in B} V_{\alpha_\mu - \frac{\alpha_a + \alpha_b}{2}} + V_{\alpha_b - \alpha_a} + V_{\alpha_a - \alpha_b}$$

For any $(I_1, \dots, I_{i-1}, A, a, b, B, K_2, \dots, K_{l-i}) \in \mathcal{I}_2$, it parameterizes a fixed map $f : \Sigma \rightarrow Fl$ described as:

- Σ nodal curve with two rational components Σ_1 and Σ_2 ;
- f maps the node to $P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, K_2, \dots, K_{l-i}; k}$;
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, K_2, \dots, K_{l-i}; k}$ and $P_{I_1, \dots, I_{i-1}, A \cup \{b\}, \{a\} \cup B, K_2, \dots, K_{l-i}; k}$ when restricted $\Sigma_i, i = 1, 2$.

In this case,

$$(58) \quad \text{Def}(\Sigma, f) = V_{I_1, \dots, I_{i-1}, A, a, b, B, K_2, \dots, K_{l-i}} + V_{A, a} + V_{b, B} + V_{2\alpha_b - 2\alpha_a}$$

For any $(I_1, \dots, I_{i-1}, A, a, \{b_1, b_2\}, B, K_2, \dots, K_{l-i}) \in \mathcal{I}_3$, it parameterizes a fixed map $f : \Sigma \rightarrow Fl$ described as:

- Σ nodal curve with two rational components Σ_1 and Σ_2 ;
- f maps the node to $P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b_1, b_2\} \cup B, K_2, \dots, K_{l-i}; k}$;
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b_1, b_2\} \cup B, K_2, \dots, K_{l-i}; k}$ and $P_{I_1, \dots, I_{i-1}, A \cup \{b_1\}, \{a, b_2\} \cup B, K_2, \dots, K_{l-i}; k}$ when restricted $\Sigma_i, i = 1, 2$.

In this case,

$$(59) \quad \text{Def}(\Sigma, f) = V_{I_1, \dots, I_{i-1}, A, a, \{b_1, b_2\}, B, K_2, \dots, K_{l-i}} + V_{A, a} + V_{\alpha_{b_1} + \alpha_{b_2} - 2\alpha_a} + V_{\alpha_{b_2} - \alpha_{b_1}} + V_{\alpha_{b_1} - \alpha_{b_2}}$$

For any $(I_1, \dots, I_{i-1}, A, \{a_1, a_2\}, b, B, K_2, \dots, K_{l-i}) \in \mathcal{I}_4$, it parameterizes a fixed map $f : \Sigma \rightarrow Fl$ described as:

- Σ nodal curve with two rational components Σ_1 and Σ_2 ;
- f maps the node to $P_{I_1, \dots, I_{i-1}, A \cup \{a_1, a_2\}, \{b\} \cup B, K_2, \dots, K_{l-i}; k}$;
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{I_1, \dots, I_{i-1}, A \cup \{a_1, a_2\}, \{b\} \cup B, K_2, \dots, K_{l-i}; k}$ and $P_{I_1, \dots, I_{i-1}, A \cup \{a_1\}, \{a_2, b\} \cup B, K_2, \dots, K_{l-i}; k}$ when restricted $\Sigma_i, i = 1, 2$.

In this case,

$$(60) \quad \text{Def}(\Sigma, f) = V_{I_1, \dots, I_{i-1}, A, \{a_1, a_2\}, b, B, K_2, \dots, K_{l-i}} + V_{\alpha_{a_1} - \alpha_{a_2}} + V_{\alpha_{a_2} - \alpha_{a_1}} + V_{b, B}$$

For any $(I_1, \dots, I_{i-1}, A, \{a_1, a_2\}, b_1, b_2, B, K_2, \dots, K_{l-i}) \in \mathcal{I}_5$, it parameterizes a fixed map $f : \Sigma \rightarrow Fl$ described as:

- Σ nodal curve with two rational components Σ_1 and Σ_2 ;
- f maps the node to $P_{I_1, \dots, I_{i-1}, A \cup \{a_1, a_2\}, \{b_1, b_2\} \cup B, K_2, \dots, K_{l-i}; k}$;
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{I_1, \dots, I_{i-1}, A \cup \{a_1, a_2\}, \{b_1, b_2\} \cup B, K_2, \dots, K_{l-i}; k}$ and $P_{I_1, \dots, I_{i-1}, A \cup \{b_1, a_2\}, \{a_1, b_2\} \cup B, K_2, \dots, K_{l-i}; k}$ when restricted $\Sigma_i, i = 1, 2$, where we assume $a_1 < a_2$.

In this case,

$$(61) \quad \text{Def}(\Sigma, f) = V_{I_1, \dots, I_{i-1}, A, \{a_1, a_2\}, \{b_1, b_2\}, B, K_2, \dots, K_{l-i}} + V_{\alpha_{a_1} - \alpha_{a_2}} + V_{\alpha_{a_2} - \alpha_{a_1}} + V_{\alpha_{b_1} - \alpha_{b_2}} + V_{\alpha_{b_2} - \alpha_{b_1}} + V_{\alpha_{b_1} + \alpha_{b_2} - \alpha_{a_1} - \alpha_{a_2}}$$

To get the contributions of each case, we count the number of positive weights in the representations (57)(58)(59)(60)(61), and then sum over the index sets $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5$ respectively as in the formula (12). Actually in this case, we do not have to do the complicated computation. In fact, we can use the fibrations $\pi_j : \mathcal{I}_j \rightarrow \mathcal{I}_{r_1, \dots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \dots, r_{l+1}}$ to reduce to the Grassmannian case. For example, using the fibration π_1 , the contribution of fixed locus indexed by \mathcal{I}_1 is:

$$(62) \quad \sum_{\substack{(I_1, \dots, I_{i-1}, J, K_2, \dots, K_{l-i}) \\ \in \mathcal{I}_{r_1, \dots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \dots, r_{l+1}}}} q^{2N_{I_1, \dots, I_{i-1}, J, K_2, \dots, K_{l-i}}} \sum_{\substack{(A, \{a, b\}, B) \\ \in \mathcal{I}_{r_i - 1, 2, r_{i+1} - 1}}} q^{2(N_{A, \{a, b\}, B} + \#\{\nu \in A \mid \frac{\alpha_a + \alpha_b}{2} > \alpha_\nu\} + \#\{\mu \in B \mid \frac{\alpha_a + \alpha_b}{2} < \alpha_\mu\} + 1)} \\ = \binom{k}{r_1, \dots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \dots, r_{l+1}} q^2 \sum_{\substack{(A, \{a, b\}, B) \\ \in \mathcal{I}_{r_i - 1, 2, r_{i+1} - 1}}} q^{2(N_{A, \{a, b\}, B} + \#\{\nu \in A \mid \frac{\alpha_a + \alpha_b}{2} > \alpha_\nu\} + \#\{\mu \in B \mid \frac{\alpha_a + \alpha_b}{2} < \alpha_\mu\} + 1)}$$

and one can recognize that the summation in the second line is the contribution of fixed locus of type \mathcal{I}_1 with the target space replaced by the Grassmannian $\text{Gr}(r_i, r_i + r_{i+1})$. We finally get the result:

$$(63) \quad \begin{aligned} P_{\mathcal{M}_0(Fl, 2\check{H}_i)}(q) &= \binom{k}{r_1, \dots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \dots, r_{l+1}}_{q^2} P_{\mathcal{M}_0(\text{Gr}(r_i, r_i + r_{i+1}), 2)}(q) \\ &= \frac{(1 - t^{r_i})(1 - t^{r_{i+1}})((1 + t^{r_i + r_{i+1}})(1 + t^3) - t(1 + t)(t^{r_i} + t^{r_{i+1}}))}{(1 - t)^2(1 - t^2)^2} P_{Fl}(q) \end{aligned}$$

where in the second equality, we use expression of $P_{\mathcal{M}_0(\text{Gr}(r_i, r_i + r_{i+1}), 2)}$ given in [14, Theorem 3.1].

4.7. Poincaré polynomial of $\mathcal{M}_0(Fl, \check{H}_i + \check{H}_{i+1})$. In this case, there are three kinds of fixed maps, which are parameterized by the sets \mathcal{I}_i , \mathcal{I}'_i and $\mathcal{A}_{i-1, 1, l-i-1}$ respectively, where

$$\mathcal{I}_i = \mathfrak{J}_{r_1, \dots, r_{i-1}, r_i - 1, 1, 1, r_{i+1} - 2, 1, 1, r_{i+1}, \dots, r_{l+1}}$$

and

$$\mathcal{I}'_i = \mathfrak{J}_{r_1, \dots, r_{i-1}, r_i - 1, 1, 1, r_{i+1} - 1, 1, r_{i+2} - 1, r_{i+3}, \dots, r_{l+1}}$$

For every $(I_1, \dots, I_{i-1}, A, a, b_1, B, b_2, c, C, K_2, \dots, K_{l-i}) \in \mathcal{I}_i$, the corresponding fixed map $f : \Sigma \rightarrow Fl$ can be described as:

- Σ is a nodal curve with two rational components Σ_1 and Σ_2 .
- $f(p) = P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b_1, b_2\} \cup B, \{c\} \cup C, K_2, \dots, K_{l-i}}$, where p is the node of Σ .
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{\dots, I_{i-1}, A \cup \{a\}, \{b_1, b_2\} \cup B, \{c\} \cup C, K_2, \dots}$ and $P_{\dots, I_{i-1}, A \cup \{b_1\}, \{a, b_2\} \cup B, \{c\} \cup C, K_2, \dots}$ when restricted to Σ_1 .
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{\dots, I_{i-1}, A \cup \{a\}, \{b_1, b_2\} \cup B, \{c\} \cup C, K_2, \dots}$ and $P_{\dots, I_{i-1}, A \cup \{a\}, \{b_1, c\} \cup B, \{b_2\} \cup C, K_2, \dots}$ when restricted to Σ_2 .

The computation of $\text{Def}(\Sigma, f)$ is similar to that in the last case, one can easily check that:

$$(64) \quad \begin{aligned} \text{Def}(\Sigma, f) &= \text{Def}(f) + \text{Def}(\Sigma) - \text{Aut}(\Sigma) \\ &= V_{I_1, \dots, I_{i-1}, A, a, b_1, B, b_2, c, C, K_2, \dots, K_{l-i}} + V_{\alpha_{b_1} - \alpha_a + \alpha_c - \alpha_{b_2}} + V_{\alpha_{b_2} - \alpha_{b_1}} \end{aligned}$$

For every $(I_1, \dots, I_{i-1}, A, a, B, b, c, C, K_2, \dots, K_{l-i}) \in \mathcal{I}'_i$, the corresponding fixed map $f : \Sigma \rightarrow Fl$ can be described as:

- Σ is a nodal curve with two rational components Σ_1 and Σ_2 .
- $f(p) = P_{I_1, \dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, \{c\} \cup C, K_2, \dots, K_{l-i}}$, where p is the node of Σ .
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{\dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, \{c\} \cup C, K_2, \dots}$ and $P_{\dots, I_{i-1}, A \cup \{b\}, \{a\} \cup B, \{c\} \cup C, K_2, \dots}$ when restricted to Σ_1 .
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{\dots, I_{i-1}, A \cup \{a\}, \{b\} \cup B, \{c\} \cup C, K_2, \dots}$ and $P_{\dots, I_{i-1}, A \cup \{a\}, \{c\} \cup B, \{b\} \cup C, K_2, \dots}$ when restricted to Σ_2 .

One can check that:

$$(65) \quad \text{Def}(\Sigma, f) = V_{I_1, \dots, I_{i-1}, A, a, B, b, c, C, K_2, \dots, K_{l-i}} + V_{b, B} + V_{\alpha_c - \alpha_a}$$

For every $(I_1, \dots, I_{i-1}, A, \{a, b\}, J, B, K_1, \dots, K_{l-i-1}) \in \mathcal{A}_{i-1, 1, l-i-1}$, the corresponding fixed map $f : \Sigma \rightarrow Fl$ can be described as:

- Σ is isomorphic to \mathbb{P}^1 .
- f is a one-to-one map from \mathbb{P}^1 to the fixed line connecting $P_{\dots, I_{i-1}, A \cup \{a\}, J, \{b\} \cup B, K_1, \dots, K_{l-i-1}}$ and $P_{\dots, I_{i-1}, A \cup \{b\}, J, \{a\} \cup B, K_1, \dots, K_{l-i-1}}$.

One can check that:

$$(66) \quad \text{Def}(\Sigma, f) = V_{I_1, \dots, I_{i-1}, A, \{a, b\}, J, B, K_1, \dots, K_{l-i-1}} + V_{J, \{a, b\}}$$

To simplify the computation, we now assume the flag is a complete flag, i.e. $r_i = 1, i = 1, \dots, l + 1$. In this special case, $\mathcal{I}_i = \emptyset$, and we only need to take summation over \mathcal{I}'_i and $\mathcal{A}_{i-1, 1, l-i-1}$. Using the fibration

$\pi_1 : \mathcal{I}'_i \rightarrow \mathcal{I}_{r_1, \dots, r_i-1, r_i+r_{i+1}+r_{i+2}, r_{i+3}, \dots, r_{l+1}}$, the contribution of \mathcal{I}'_i is:

$$\begin{aligned}
 & \left(\begin{matrix} k \\ 1, \dots, 1, 3, 1, \dots, 1 \end{matrix} \right)_{q^2} \sum_{(a,b,c) \in \mathcal{I}_{1,1,1}} q^{2(N_{a,b,c} + \delta(c-a))} \\
 &= \left(\begin{matrix} k \\ 1, \dots, 1, 3, 1, \dots, 1 \end{matrix} \right)_{q^2} (1 + 2t + 3t^3 + t^4) \\
 (67) \quad &= \frac{1 + 2t + 3t^3 + t^4}{(1+t)(1+t+t^2)} P_{Fl}(q)
 \end{aligned}$$

Similarly the contribution of $\mathcal{A}_{i-1,1,l-i-1}$ is:

$$(68) \quad \left(\begin{matrix} k \\ 1, \dots, 1, 3, 1, \dots, 1 \end{matrix} \right)_{q^2} \binom{3}{2} t^2 = \frac{3t^2}{(1+t)(1+t+t^2)} P_{Fl}(q)$$

In sum, we have:

$$(69) \quad P_{\mathcal{M}_0(Fl, \tilde{H}_i + \tilde{H}_{i+1})}(q) = \frac{1 + 2t + 3t^2 + 3t^3 + t^4}{(1+t)(1+t+t^2)} P_{Fl}(q)$$

REFERENCES

- [1] Shishir Agrawal. The euler characteristic of the moduli space of stable maps into a grassmannian. Undergraduate thesis, University of California, San Diego, 2011.
- [2] Michael F Atiyah and Isadore M Singer. The index of elliptic operators: III. *The Annals of Mathematics*, 87(3):546–604, 1968.
- [3] K. Behrend. Cohomology of stacks. *Advances in Mathematics*, 198:583–622, 2005.
- [4] A. Bialynicki-Birula. Some theorems on actions of algebraic groups. *Annals of Mathematics*, 98(3):480–497, 1973.
- [5] A. Bialynicki-Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bull.acad.polon.sci.sr.sci.math.astronom.phys*, 24(9):667–674, 1976.
- [6] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, pages 33–85. Springer, 2005.
- [7] James B. Carrell. *Torus Actions and Cohomology*. Springer Berlin Heidelberg, 2002.
- [8] Linda Chen. Poincare polynomials of hyperquot schemes. *Mathematische Annalen*, 319(2):pgs. 235–252, 2000.
- [9] Gregory Edwards. On genus zero stable maps to the flag variety. Undergraduate thesis, University of California, San Diego, 2013.
- [10] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. *Arxiv Cornell University Library*, 1996.
- [11] Kentaro Hori. *Mirror symmetry*, volume 1. American Mathematical Soc., 2003.
- [12] Victor Kac and Pokman Cheung. *Quantum calculus*. Springer Science & Business Media, 2002.
- [13] Yuri I. Manin. Stable maps of genus zero to flag spaces. *Topological Methods in Nonlinear Analysis*, (2):207–217, 1998.
- [14] Alberto López Martín. Poincaré polynomials of stable map spaces to Grassmannians. In *Rendiconti del Seminario matematico della Università di Padova*, pages 193–208, 2013.
- [15] Dragos Oprea. Tautological classes on the moduli spaces of stable maps to pr via torus actions. *Advances in Mathematics*, 207(2):661–690, 2006.
- [16] Jonathan Skowera. Bialynicki-birula decomposition of deligne-mumford stacks. *Proceedings of the American Mathematical Society*, 141(6):1933–1937, 2013.
- [17] Stein Arild Strømme. *On parametrized rational curves in Grassmann varieties*. Springer Berlin Heidelberg, 1987.